

# Optimal generation of entanglement under local control

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We study the optimal generation of entanglement between two qubits subject to local unitary control. With the only assumptions of linear control and unitary dynamics, by means of a numerical protocol based on the variational approach (Pontryagin's Minimum Principle), we evaluate the optimal control strategy leading to the maximal achievable entanglement in an arbitrary interaction time, taking into account the energy cost associated to the controls. In our model we can arbitrarily choose the relative weight between a large entanglement and a small energy cost.

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*Introduction.*— The building block for the implementation of quantum technologies is a pair of interacting two-level systems (qubits), whose evolution can be affected by external actions. The basic operations to be accomplished through this system are the storage and manipulation of encoded information. The use of this prototype of physical apparatus is motivated by its quantum nature, that is the existence of some peculiar properties not exhibited by classical systems. Among them, the quantum correlation called *entanglement* plays a prominent role in the realization of outperforming protocols.

The standard setting for the manipulation of a pair of qubits is the so-called *Local Unitary control* [1, 2]. In this framework, neglecting the influence of the external environment, the state system is represented by the unit norm complex vector  $|\psi(t)\rangle$ , satisfying the Schrödinger equation

$$|\dot{\psi}(t)\rangle = -i H_T(u_1(t), u_2(t)) |\psi(t)\rangle, \quad (1)$$

where  $H_T(u_1(t), u_2(t)) = H_1(u_1(t)) + H_2(u_2(t)) + H_I$  is the total Hamiltonian. The local contributions  $H_i(u_i(t))$  ( $i = 1, 2$ ) can be modified by means of external actions, represented by the control functions  $u_1(t)$  and  $u_2(t)$ , and  $H_I$  is the uncontrollable interaction term, responsible for the entanglement created in the system. We find convenient to express the initial state  $|\psi_0\rangle = |\psi(0)\rangle$  according to the Schmidt decomposition,

$$|\psi_0\rangle = \sqrt{P} |\varphi\rangle \otimes |\chi\rangle + \sqrt{1-P} |\varphi^\perp\rangle \otimes |\chi^\perp\rangle, \quad (2)$$

where  $P \in [0, 1]$  and  $\langle \varphi | \varphi^\perp \rangle = \langle \chi | \chi^\perp \rangle = 0$ . If there is not initial correlation between the two qubits,  $P = 0$  or  $P = 1$  and  $|\psi_0\rangle$  is a product state. Conversely, a maximally entangled state (Bell state) corresponds to  $P = 0.5$ . As a measure of entanglement we introduce the *concurrence*  $C$  defined as

$$C(t) = |\langle \psi(t) | \sigma_y \otimes \sigma_y | \psi(t) \rangle^*|, \quad (3)$$

assuming values in the interval  $[0, 1]$ , vanishing for uncorrelated states, and reaching its maximum for maximally

entangled states. This quantity satisfies all the properties of an entanglement monotone [3].

Because of its fundamental relevance, several issues regarding the entanglement generation in the system (1) have been addressed in the past years, as well as the problem of generation of nonlocal gates. In [4], Dür *et al.* characterized the capability of creating entanglement for a generic interaction  $H_I$ . In particular, they provided a strategy to minimize the time of generation of entanglement through arbitrarily fast local control, and considered the impact of ancillas. Moreover, these authors proved that an initial amount of entanglement leads to a more efficient production of entanglement. In a different context, Kraus and Cirac expressed the maximal attainable entanglement for an arbitrary unitary operator, as well as the corresponding initial factorized state [5]. The generation of entanglement implemented via local measurements has been considered in [6].

Time-optimality has been further considered in the context of the simulation of a quantum gate, by introducing the *Interaction Cost*, that is the minimal time to perform a gate using local operations [7, 8, 9].

While time-optimal procedures are fundamental for the implementation of efficient computational architectures, they usually ask for impulsive controls (instantaneous local manipulations of arbitrary strength), and their operational cost (the energy loss associated to the local controls) is not accounted for. Moreover, for some physical apparatus it could be difficult to precisely set the optimal interaction time, and a predetermined time could be preferable.

With these motivations in mind, in this letter we describe a control theoretical approach accounting for the aforementioned cost and considering an arbitrary interaction time. Under the assumption of linear control with  $H_i(u_i(t)) = u_i(t) H_i$ ,  $i = 1, 2$ , and using a numerical protocol based on the variational method, we evaluate the best control strategy, that is the optimal control functions  $u_i(t)$  driving an arbitrary initial state  $|\psi_0\rangle$  as

close as possible to a maximally entangled state. This is not the most general form for linear control, however it highly reduces the computational complexity of the problem while preserving most of its relevant features. This approach represents a novelty with respect to previous treatments and it complements them. We will mainly refer to [4] for a comparison.

Optimal control methods have been initially used in quantum mechanics for the control of molecular dynamics [10]. Recent applications of these techniques in quantum information are in particular aimed to determine the optimal gate generation [11, 12, 13] and the optimal evolution and state transfer (e.g. see [14, 15]).

*The computational procedure.*— We find the optimal control strategies  $u_i(t)$  using an iterative procedure based on the variational approach known as *Pontryagin Minimum Principle*. The computational tools employed to derive the results presented in this work can be used as well for the solution of more general optimal control problems. In fact, different performance measures as well as more complicate system dynamics (for example in the presence of irreversibility and dissipation) can be imposed. A detailed discussion of the protocol, of its performance, and of further applications is out of the scope of this letter, and it will be presented in a forthcoming paper. However, for sake of completeness, we summarize here the basic ideas underlying the procedure, without entering into details.

Consider a system described by the state  $\mathbf{x}(t) : \mathbb{R} \rightarrow \mathbb{R}^n$  whose dynamics and initial conditions are given by

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)), \quad \mathbf{x}(0) = \mathbf{x}_0, \quad (4)$$

where  $\mathbf{u}(t) : \mathbb{R} \rightarrow \mathbb{R}^m$  is a vector of control functions and  $\mathbf{f}(\mathbf{x}, \mathbf{u}) : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}^n$  is a vector field. Assume that  $\mathbf{u}(t)$  has to be chosen such that the cost functional (or performance measure)

$$J(\mathbf{u}(t)) = \Phi(\mathbf{x}(\tau)) + \alpha \int_0^\tau \mathcal{L}(\mathbf{x}(t), \mathbf{u}(t)) dt \quad (5)$$

is minimal, where  $\tau$  is the fixed final time and  $\Phi(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $\mathcal{L}(\mathbf{x}, \mathbf{u}) : \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}$  are arbitrary functions. Notice that the cost  $J$  contains a final-time term and an integral contribution (that we shall later denote by  $I(\tau)$ ), and the real coefficient  $\alpha$  expresses the relative weight of them. Define the *optimal control Hamiltonian* as

$$\mathcal{H}(\mathbf{x}(t), \mathbf{p}(t), \mathbf{u}(t)) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \cdot \mathbf{p}(t) + \mathcal{L}(\mathbf{x}(t), \mathbf{u}(t)), \quad (6)$$

where  $\cdot$  is the inner product, and  $\mathbf{p}(t) : \mathbb{R} \rightarrow \mathbb{R}^n$  is an auxiliary variable conjugate to  $\mathbf{x}(t)$ , often called costate, whose dynamics is

$$\dot{\mathbf{p}}(t) = -\nabla_{\mathbf{x}} \mathcal{H}(\mathbf{x}(t), \mathbf{p}(t), \mathbf{u}(t)) \quad (7)$$

with final condition

$$\mathbf{p}(\tau) = -\nabla_{\mathbf{x}} \Phi(\mathbf{x}(\tau)). \quad (8)$$

Then the Pontryagin Minimum Principle can be stated as follows.

**Proposition** If we denote by  $\mathbf{u}'(t)$  the optimal control strategy, and by  $\mathbf{x}'(t)$  and  $\mathbf{p}'(t)$  the corresponding optimal state and costate trajectories, then, for all  $t \in [0, \tau]$ ,

$$\mathcal{H}(\mathbf{x}'(t), \mathbf{p}'(t), \mathbf{u}'(t)) \leq \mathcal{H}(\mathbf{x}'(t), \mathbf{p}'(t), \mathbf{u}(t)). \quad (9)$$

Consequently,

$$\nabla_{\mathbf{u}} \mathcal{H}(\mathbf{x}'(t), \mathbf{p}'(t), \mathbf{u}'(t)) = 0. \quad (10)$$

While a more general formulation can be given to this principle (in particular a non-fixed final time  $\tau$  can be considered), this approach is all we need for our purposes. For more details and for the proof of the principle, see [16] and [17] (mainly focusing on quantum mechanical applications).

In our case, we find convenient to represent the state vector  $|\psi\rangle$  in the computational basis  $\{|e_i(t)\rangle, i = 1, \dots, 4\}$  given by tensor products of eigenvectors of  $\sigma_z$ ,

$$|\psi(t)\rangle = \sum_{i=1}^4 \psi_i(t) |e_i\rangle, \quad \psi_i(t) = \langle e_i | \psi(t) \rangle. \quad (11)$$

and  $\mathbf{x}(t) = (\text{Re } \psi_i(t), \text{Im } \psi_i(t); i = 1, \dots, 4)^T$ , where  $T$  means transposition. Moreover,  $\mathbf{u}(t) = (u_1(t), u_2(t))^T$ , and the vector field  $\mathbf{f}$  is linear in both  $\mathbf{x}$  and  $\mathbf{u}$ . The final-time contribution to the cost function is the deviation of the final entanglement from its maximal attainable value,

$$\Phi(\mathbf{x}(\tau)) = 1 - C(\tau) \quad (12)$$

whereas the integral part measures the energy loss, assumed to be proportional to the squared norm of  $\mathbf{u}$ ,

$$\mathcal{L}(\mathbf{u}(t)) = u_1^2(t) + u_2^2(t), \quad (13)$$

the prototype of energy cost for a nuclear spin driven by a magnetic field.

Our protocol consists of an iteration in which, after solving (4) and (7) with boundary condition (8), the controls are redefined step by step in order to fit the condition (10). The procedure starts with arbitrary trial functions  $u_i(t)$  and it stops when a predetermined accuracy level is reached.

*Discussion of numerical results.*— The optimal time strategy described in [4] is based on the maximization of the entanglement rate at every time. This procedure leads to a vector  $|\psi(t)\rangle$  whose (time-dependent) Schmidt coefficient is given by

$$P(t) = \sin^2(h_{max}t + \phi_0), \quad (14)$$

where  $P(0) = P = \sin^2 \phi_0$ . The *entanglement capability*  $h_{max}$  measures the ability of the interaction to produce entanglement, its definition and expression in terms of

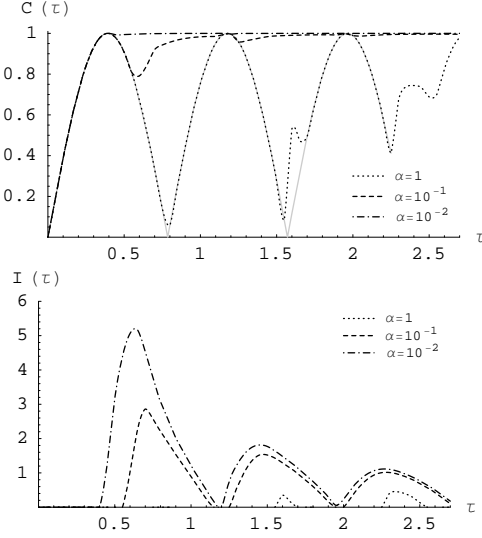


FIG. 1: Dependence on  $\tau$  of the final entanglement  $C(\tau)$  and the energy cost  $I(\tau)$  for several values of the weight parameter  $\alpha$ . The initial state is an optimal state for the Heisenberg Hamiltonian, with  $P = 0$ ,  $|\varphi\rangle = |\uparrow\rangle_z$ , and  $|\chi\rangle = |\downarrow\rangle_z$ .

the singular values of  $H_I$  are given in [4]. The optimal time  $\tau_{opt}$  is defined as the smallest time such that  $P(t) = 0.5$ . Equation (14) defines the steepest entanglement growth, therefore it represents the upper limit for the production of entanglement up to the optimal time.

Our protocol works for arbitrary Hamiltonian terms and initial states. In order to illustrate the main results of this work we need to fix them. Because of its relevance, we consider the Heisenberg interaction  $H_I = \sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z$  (with  $h_{max} = 2$  and  $\tau_{opt} = \pi/8$ ),  $H_1 = \sigma_x \otimes \mathbb{I}$ , and  $H_2 = \mathbb{I} \otimes \sigma_x$ .

In Fig. 1 and 2 the dependence on  $\tau$  of the final entanglement  $C(\tau)$  and the energy cost  $I(\tau)$  (the integral in (5)) is shown for several values of  $\alpha$ , for the optimal strategies. The two figures correspond to different uncorrelated initial states ( $P = 0$ ):  $|\varphi\rangle = |\uparrow\rangle_z$ ,  $|\chi\rangle = |\downarrow\rangle_z$  and  $|\varphi\rangle = |\downarrow\rangle_y$ ,  $|\chi\rangle = |\downarrow\rangle_z$  respectively ( $|\uparrow\rangle_i$  and  $|\downarrow\rangle_i$  denote the  $+1$  and  $-1$  eigenvectors of the Pauli matrix  $\sigma_i$ ,  $i = x, y, z$ ).

The first state is an optimal initial state for the Heisenberg interaction [4, 5], then it evolves in the optimal time to a maximally entangled state without local actions. This is apparent from Fig. 1: all the represented curves for  $C(\tau)$  have the same behavior for  $\tau \leq \tau_{opt}$ , independently of  $\alpha$ . These patterns fit the entanglement evolution associated to (14), represented by the grey line, since the system is driven by  $H_I$  along the optimal trajectory. Therefore, the energy cost associated to these paths vanishes. For  $\tau > \tau_{opt}$  the parameter  $\alpha$  becomes relevant, since a local action is necessary to maximize the final entanglement. We observe that the goal  $C(\tau) = 1$  is

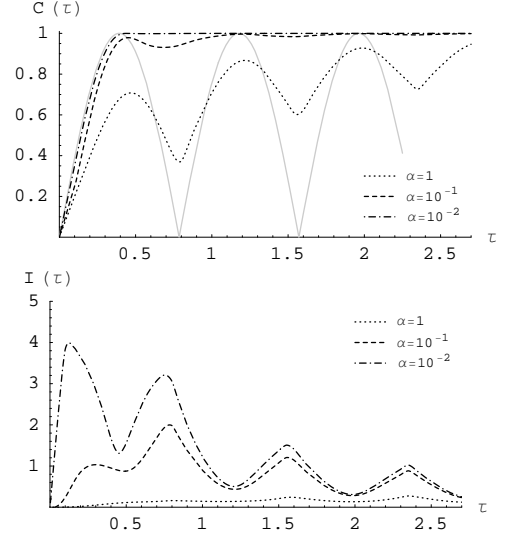


FIG. 2: Dependence on  $\tau$  of the final entanglement  $C(\tau)$  and the energy cost  $I(\tau)$  for several values of the merit function  $\alpha$ . The initial state is given by  $P = 0$ ,  $|\varphi\rangle = |\downarrow\rangle_y$ , and  $|\chi\rangle = |\downarrow\rangle_z$ .

approached as  $\alpha$  is decreased, that is a higher cost is tolerated, and that, for every  $\alpha$ ,  $C(\tau) \rightarrow 1$  when  $\tau$  increases. The peaks of  $I(\tau)$  correspond to the valleys associated to (14), where a stronger local control is needed, and the magnitude of these peaks decreases with  $\tau$ .

In Fig. 2 a similar analysis is presented for a particular non-optimal state. In this case, local manipulations are needed even for  $\tau \leq \tau_{opt}$  since the state has to be adapted in order to fully exploit the entangling capability of the interaction. This is apparent in the presented plots, where the growth of  $C(\tau)$  is steeper with  $\alpha$  smaller, and eventually it approaches the optimal curve associated to (14). Correspondingly, there is a relevant initial contribution to  $I(\tau)$ . Notice that for an arbitrary non-optimal state, different patterns could be found for  $\alpha$  large, in

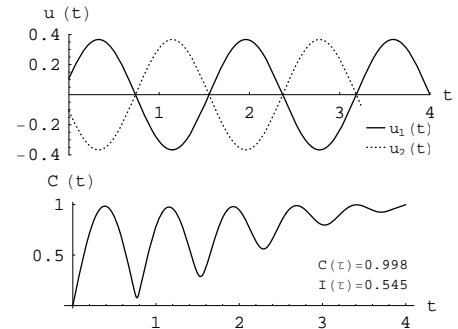


FIG. 3: Optimal strategies  $u_1(t)$  and  $u_2(t)$ , and corresponding entanglement  $C(t)$ , for the optimal initial state, and  $\tau = 4$ . We have chosen  $\alpha = 10^{-1}$ .

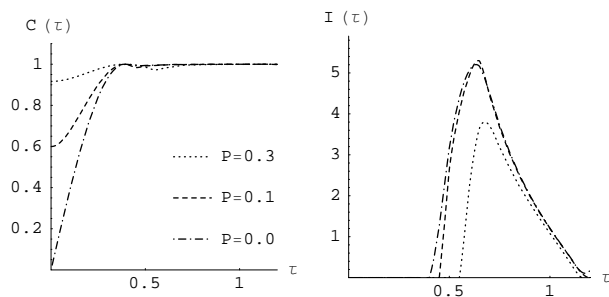


FIG. 4: Dependence on  $\tau$  of the final entanglement  $C(\tau)$  and the energy cost  $I(\tau)$  for several values of the initial Schmidt coefficient  $P$ . The initial state is defined by  $|\varphi\rangle = |\uparrow\rangle_z$ , and  $|\chi\rangle = |\downarrow\rangle_z$ . We have chosen  $\alpha = 10^{-2}$ .

particular the steepest growth of  $C(\tau)$  could be slower than the optimal one, and the plateau of  $C(\tau)$  could be (even significantly) below 1. This is due to the particular choice of  $H_1$  and  $H_2$ , that could be inappropriate for the initial state considered.

The optimal control strategy can be fixed in line with these considerations. Given the interaction time  $\tau$ , the choice of the coefficient  $\alpha$  represents a compromise between magnification of  $C(\tau)$  and reduction of  $I(\tau)$ . The optimal strategies  $u_1(t)$  and  $u_2(t)$  match the requests on  $\tau$ ,  $C(\tau)$  and  $I(\tau)$ . An example is provided in Fig. 3, with the Hamiltonian terms and the optimal initial state previously introduced, and  $\tau = 4$ .

Oscillating controls with modulated amplitude are usually obtained when  $\tau$  exceeds the optimal time. When the energy cost is a relevant factor, impulsive controls are less efficient than controls distributed over time. They become the optimal strategy when  $\tau < \tau_{opt}$ , if a large energy expense is accepted. The function  $C(t)$  does not in general stabilize around its maximum (as in the example presented here), however the amplitude of oscillations usually decreases as  $t \rightarrow \tau$ .

We have also considered initial states with a non-vanishing entanglement. Some plots are shown in Fig. 4 with  $|\varphi\rangle = |\uparrow\rangle_z$ ,  $|\chi\rangle = |\downarrow\rangle_z$  and  $P \neq 0$ . As intuition suggests, the cost  $I(\tau)$  is usually smaller for correlated initial states.

**Conclusions.**— We have considered a variational approach for the solution of optimal control problems involving two qubits. By accounting for the energy cost associated to the manipulations of the system, we are able to find the optimal strategies to be used in order to drive the system.

In this letter, we have described the entanglement generation for systems without interaction with the external environment, driven by local unitary control. Our numerical analysis is consistent with previous results, in particular it reproduces the optimal entanglement growth and the corresponding minimal time. Moreover, in our approach the interaction time can be arbitrarily chosen

(for example, it can be a fixed instrumental time or a predetermined operational time), it is not fixed by the interaction. From this point of view, our protocol complements the existing methods for the generation of entanglement, and it is of interest whenever a non-optimal interaction time is preferred or the energy cost associated to the controls has to be taken into account.

Using standard Hamiltonian terms and particular initial states, we have studied the relations among the relevant quantities, and provided some examples. We have found that a large interaction time is usually preferred for the reduction of the energy cost, without decreasing the efficiency of the entanglement production. From this point of view, controls spread over time are more convenient with respect to impulsive controls.

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